Abstract. We investigate when and how the diagonal of a space $X$ can be separated from any closed subset of the square $X \times X$ that lies off the diagonal. Several examples are given to help illuminate these properties of diagonal separation and distinguish between them.

1. Introduction

In this paper we continue to study the diagonal separation properties introduced in [Har] and in [Buz]. To discuss what has been done and what will be done let us start with the definitions:

**Definition of $\Delta$-normality.** [Har] A space $X$ is $\Delta$-normal if for every $A \subset X^2 \setminus \Delta_X$ closed in $X^2$ there exist disjoint open $U$ and $V$ in $X^2$ such that $A \subset U$ and $\Delta_X \subset V$.

**Definition of functional $\Delta$-normality.** [Buz] A space $X$ is functionally $\Delta$-normal if for every $A \subset X^2 \setminus \Delta_X$ closed in $X^2$ there exists a continuous function $f : X^2 \to [0,1]$ such that $f(A) = \{1\}$ and $f(\Delta_X) = \{0\}$.

**Definition of $\Delta$-paracompactness.** [Buz] A space $X$ is $\Delta$-paracompact if for every $A \subset X^2 \setminus \Delta_X$ closed in $X^2$ there exists a locally finite open cover $U$ of $X$ such that $\bigcup \{U \times U : U \in U\}$ does not meet $A$.

**Definition of regular $\Delta$-paracompactness.** [Buz] A space $X$ is regular $\Delta$-paracompact if for every $A \subset X^2 \setminus \Delta_X$ closed in $X^2$ there exists a locally finite open cover $U$ of $X$ such that $\bigcup \{U \times U : U \in U\}$ does not meet $A$.

**Definition functional $\Delta$-paracompactness.** [Buz] A space $X$ is functionally $\Delta$-paracompact if for every $A \subset X^2 \setminus \Delta_X$ closed in $X^2$ there exists a locally finite cover $U$ of $X$ by functionally open sets such that $\bigcup \{U \times U : U \in U\}$ does not meet $A$.

Recently, K.P. Hart communicated to the authors a proof of equivalence of functional $\Delta$-paracompactness and divisibility. Recall that a space $X$ is divisible [Csa] if for every open set $U$ containing $\Delta_X$ there exists an open set $V$ containing $\Delta_X$ such that $V \circ V \subset U$, where $V \circ V = \{(x,z) : (x,y) \in V \text{ and } (y,z) \in V, \text{ for some } y\}$.

One of natural problems is to distinguish the above properties between themselves as well as from normality in the class of Tychonov spaces. Some steps in this direction were made in [Har] and in [Buz]. It is known, for example, that functional $\Delta$-paracompactness implies all other properties [Buz]. In this paper we...
will almost nail the problem down leaving only three implications (quite intriguing ones) unanswered. In addition we find another class of spaces that have all the above properties. In Section 3 we start investigating the behavior of the properties under continuous maps leaving more questions than answers. In notation and terminology we will follow [Eng]. All spaces are assumed Tychonov.

2. Distinguishing the Properties

The properties in question cannot be distinguished in the class of paracompact spaces or generalized ordered spaces simply because every space in this class has all of them [Buz]. Another class of spaces that possess all of the properties is given by our first result. To prove it, we will use the following theorem of Balogh and Rudin [BalRud]:

Every open cover $U$ of a monotonically normal space $X$ has a $\sigma$-disjoint open partial refinement $V$ such that $X \setminus \bigcup V$ is the union of a discrete family of closed subspaces, each homeomorphic to some stationary subset of a regular uncountable cardinal.

Theorem 2.1. Let $X$ be first-countable, countably compact, and monotonically normal. Then $X$ is functionally $\Delta$-paracompact.

Proof. Let $A$ be a closed subset of $X \times X$ off the diagonal. Define $P$ as follows:

1. $x \in P$ if $x \in \beta X \setminus X$; and
2. There exist an uncountable regular cardinal $\tau_x$, a stationary subset $S_x \subset \tau_x$, and a closed subset $T_x = \{x_\alpha : \alpha \in S_x\}$ of $X$ homeomorphic to $S_x$ under correspondence $x_\alpha \leftrightarrow \alpha$ such that $x$ is the complete accumulation point for $T_x$ in $\beta X$.

In what follows by $p(1)$ and $p(2)$ we denote the first and second coordinates of $p \in X \times X$. We will often use the fact that the projections of $X \times X$ onto coordinate axes are closed maps, which is guaranteed by countable compactness and first-countability of $X$. The strategy of the proof is completely contained in the statement of Claim 5 and a short proof after that.

Claim 1: Let $x \in P$. Then there exists $\lambda < \tau_x$ such that for any $a \in A$ either $a(1) \in \{x_\alpha \in T_x : \alpha > \lambda\}$ or $a(2) \in \{x_\alpha \in T_x : \alpha > \lambda\}$ fails.

If the claim fails we can select a strictly increasing sequence $\{\gamma_n\}_n$ of ordinals of $S_x$ and a sequence $\{a_n\}_n$ of elements of $A$ such that $a_n(1), a_n(2) \in \{x_\alpha \in T_x : \gamma_n < \alpha < \gamma_{n+1}\}$. By countable compactness and first-countability of $X$, an infinite subsequence of $\{a_n\}_n$ converges to some point $a \in A$. By our choice of original sequences, we have $a(1) = a(2) = x_\gamma$, where $\gamma = \lim_{n \to \infty} \gamma_n$. This means that $a \in \Delta_X$, contradicting the fact that $A$ misses the diagonal. The claim is proved.

Claim 2: Let $x \in P$. Then there exist $\lambda < \tau_x$ and a functionally open neighborhood $U$ of $\text{Cl}_{\beta X}(\{x_\alpha \in T_x : \alpha > \lambda\})$ in $\beta X$ such that $a(2) \notin U$ whenever $a \in A$ and $a(1) \in \{x_\alpha \in T_x : \alpha > \lambda\}$.

Let $\lambda$ be as in Claim 1 and let $\mathcal{O}$ be the family of functionally open neighborhoods of $\text{Cl}_{\beta X}(\{x_\alpha \in T_x : \alpha > \lambda\})$ in $\beta X$. Let us show that one of
the elements of $O$ together with $\lambda$ meet the conclusion of Claim 2. Assume the contrary. Then for any $O \in O$ we can select $a_O \in A$ such that $a_O(1) \in \{x_\alpha \in T_\alpha : \alpha > \lambda\}$ and $a_O(2) \in O$. Since $X$ is normal $\{x_\alpha \in T_\alpha : \alpha > \lambda\}$ meets $\{a_O(2) : O \in O\}$. Pick $y$ in the intersection. Since projections are closed, we can find $a \in \{a_O : O \in O\}$ such that $a(2) = y$. By continuity of projections, $a(1) \in \{x_\alpha \in T_\alpha : \alpha > \lambda\}$. Thus, $a(1), a(2) \in \{x_\alpha \in T_\alpha : \alpha > \lambda\}$, contradicting the requirement that $\lambda$ meets the conclusion of Claim 1. The claim is proved.

**Claim 3:** Let $x \in P$. Then there exist $\alpha < \beta_x$ and a functionally open neighborhood $W$ of $\text{Cl}_{\beta X}(\{x_\alpha \in T_\alpha : \alpha > \lambda\})$ in $\beta X$ such that for any $a \in A$ either $a(1) \notin W$ or $a(2) \in W$ fails.

It is clear that Claim 2 holds if we replace $a(1)$ with $a(2)$ in the conclusion. Therefore there exist $\alpha < \beta_x$ and an open neighborhood $U$ of $\text{Cl}_{\beta X}(\{x_\alpha \in T_\alpha : \alpha > \lambda\})$ in $\beta X$ that meet the conclusions of both versions of Claim 2. Let $O$ be the family of functionally open neighborhoods of $\text{Cl}_{\beta X}(\{x_\alpha \in T_\alpha : \alpha > \lambda\})$ in $\beta X$ whose closures are subsets of $U$.

Assume the conclusion of the claim fails. Then for every $O \in O$ we can find $a_O \in A$ with $a_O(1), a_O(2) \in O$. Since $X$ is normal $\{x_\alpha \in T_\alpha : \alpha > \lambda\}$ meets $\{a_O(1) : O \in O\}$. Pick $y$ in the intersection. Since projections are closed we can find $a \in \{a_O : O \in O\}$ such that $a(1) = y$. Since $\text{Cl}_{\beta X}(O) \subset U$ for every $O \in O$, we have $a(2) \in U$. Thus, $a(1) \in \{x_\alpha \in T_\alpha : \alpha > \lambda\}$ and $a(2) \in U$, contradicting the requirement that $\{\lambda, U\}$ meets the conclusion of Claim 1. The claim is proved.

**Claim 4:** Let $x \in P$. Then there exists a functionally open neighborhood $W$ of $x$ in $\beta X$ such that $(W \times W) \cap (X \times X)$ misses $A$.

Clearly $W$ from Claim 3 is as desired.

**Claim 5:** Let $T$ be a closed subset of $X$ homeomorphic to a stationary subset of some uncountable regular cardinal. Then there exists a finite functionally open cover $U$ of $T$ such that $\bigcup\{U \times U : U \in U\}$ misses $A$.

Since $X$ is Tychonov, for every $x \in T$ we can find a functionally open neighborhood $U_x$ of $x$ in $\beta X$ such that $(U_x \times U_x) \cap (X \times X)$ misses $A$. If $x \in \beta T \setminus T$ then $x \in P$, so fix a functionally open $U_x$ that meets the conclusion of Claim 4. The family $\{U_x : x \in \beta T\}$ is an open cover of $\beta T$. Since $\beta T$ is compact the cover contains a finite subcover $U$. Clearly $\{U \cap X : U \in U\}$ is as desired.

We are at the final stage of our proof. Let $O$ be a functionally open cover of $X$ such that $\bigcup\{O \times O : O \in O\}$ misses $A$. By Rudin-Balogh theorem there exists a $\sigma$-disjoint open family $V$ such that every element of $V$ is a subset of an element of $O$ and $X \setminus \bigcup V$ is the union of a discrete family $D$ of closed subsets of $X$ each of which is homeomorphic to a stationary subset of some uncountable regular cardinal. Since $X$ is countably compact $D$ is finite. For each $T \in D$ fix an open cover $U_T$ that meets the conclusion of Claim 5. Since $Y = X \setminus \bigcup\{U : U \in U_T, T \in D\}$ is countably compact and $\mathcal{V}_Y = \{V \cap Y : V \in \mathcal{V}\}$ is $\sigma$-disjoint family of open sets that covers $Y$, the family $\mathcal{V}_Y$ is finite. Therefore there exists a finite subfamily $U$ of $O$ that covers $Y$. Thus, $W = U \cup \{U_T : T \in D\}$ is finite functionally open cover of $X$ such that $\bigcup\{W \times W : W \in W\}$ misses $A$. The theorem is proved. □
Theorem 2.1 and the fact that GO-spaces have all the properties prompt the question whether monotone normality in \(X\) suffices for the desired conclusion. However, it does not. In [Har, Example 3.1], it is shown that one classical example of a monotonically normal, hereditarily countably paracompact space is not not even \(\Delta\)-normal, and therefore, not functionally \(\Delta\)-paracompact. The example, however, has a huge closed discrete subspace, which motivates the following question.

**Question 2.2.** Let \(X\) be monotonically normal and have countable extent. Does \(X\) have any of the \(\Delta\)-separation properties? What if \(X\) is countably compact?

Our first goal in distinguishing the properties is to show that functional \(\Delta\)-normality does not imply \(\Delta\)-paracompactness. For this we start by establishing that if \(X\) is a \(\Delta\)-paracompact space then the cardinality of every closed discrete subset of \(X\) cannot exceed the density of \(X\). For this we call a space \(X\) strongly collectionwise Hausdorff if for every closed discrete subset \(A \subseteq X\) there exists a discrete collection \(\{ U_a : a \in A \}\) of open sets such that \(U_a \cap A = \{a\}\) for all \(a \in A\).

**Theorem 2.3.** Let \(X\) be \(\Delta\)-paracompact. Then \(X\) is strongly collectionwise Hausdorff.

*Proof.* Let \(F\) be a closed discrete subset of \(X\). By regularity, for each \(x \in F\) we can fix an open neighborhood \(O_x\) of \(x\) such that \(\overline{O_x}\) does not meet \(F \setminus \{x\}\). Put \(O^* = X \setminus F\). The family \(O = \{O^*\} \cup \{O_x : x \in F\}\) is an open cover of \(X\). Therefore, the set \(A = (X \times X) \setminus \bigcup \{ O \times O : O \in O \}\) is a closed subset of \(X \times X\) that misses the diagonal. Since \(X\) is \(\Delta\)-paracompact there exists a locally finite open cover \(U\) of \(X\) such that \(\bigcup\{ U \times U : U \in U\}\) misses \(A\).

**Claim:** If \(x, y \in F\) are distinct and \(x \in U \in U\) then \(y \notin U\).

To prove the claim, assume the conclusion does not hold. Since \(y \notin \overline{O_x}\) and \(y \in U\) we conclude that there exists \(z \in U \setminus \overline{O_x}\). Since among all elements of \(O\) only \(O_x\) contains \(z\) we have \(\langle x, z \rangle \in A\). At the same time \(\langle x, z \rangle \in U \times U\), contradicting the choice of \(U\). The claim is proved.

For each \(x \in F\) let \(U_x\) consist of all elements of \(U\) that contain \(x\) and \(V_x\) consist of all elements of \(U\) that contain at least one element of \(F \setminus \{x\}\).

Let us show that \(\bigcup V_x\) does not contain \(x\). For this observe first that \(\bigcup V_x = \bigcup \{ V : V \in V_x \}\). This follows from local finiteness of \(U\). By the Claim and the definition of \(V_x\), we have \(V_x\) does not contain \(x\), for every \(V \in V_x\), so \(\bigcup V_x\) does not contain \(x\).

Since \(X\) is regular, there exists an open neighborhood \(W_x\) of \(F\) such that \(\overline{W_x}\) is a subset of \(\bigcup U_x\) and does not meet \(\bigcup V_x\). Let us show that \(\overline{W_x} \cap \overline{W_y} = \emptyset\) for distinct \(x, y \in F\). By the definitions, we have \(U_y \subseteq V_x\). By our choice, \(\overline{W_x}\) does not meet \(\bigcup V_x\) while \(\overline{W_y}\) is a subset of \(\bigcup V_x\).

It suffices to show now that the family \(\{ W_x : x \in F\}\) is locally finite. Fix \(y \in X\). Since \(U\) is locally finite there exists an open \(U_y\) that contains \(y\) and meets only finitely many members of \(U\). Since \(W_x\) is a subset of \(\bigcup U_x\) for each \(x \in F\) we only need to show that \(U_y\) meets \(\bigcup U_x\) for finitely many \(x \in F\) only. Assume the contrary and let \(U_y\) meet \(\bigcup U_x\) for every \(x \in S \subseteq F\), where \(S\) is infinite. Since \(U\) is locally finite and each \(U_x\) is a finite subset of \(U\) we can find \(U \subseteq U\) and distinct \(a, b \in S\) such that \(U\) is a member of both \(U_a\) and \(U_b\). By the definition of \(U_x\), we have \(a \in U\) and \(b \in U\), contradicting the Claim. \(\square\)
Corollary 2.4. Let $X$ be $\Delta$-paracompact. Then the cardinality of any closed discrete subset of $X$ cannot exceed the density of $X$.

It was shown in [Buz] that $\Delta$-paracompactness does not imply $\Delta$-normality. The preceding corollary leads to the following examples.

Example 2.5.
(a) There exists an example of a functionally $\Delta$-normal space that is not $\Delta$-paracompact.
(b) There exists a consistent example of a normal and functionally $\Delta$-normal space that is not $\Delta$-paracompact.

Proof. For (a) we can use the Heath V-space $H$ described below before the proof of Lemma 2.9. It is shown that this space is functionally $\Delta$-normal; it cannot be $\Delta$-paracompact since it is not collectionwise Hausdorff.

For (b) let $X$ be a separable space that contains an uncountable discrete subset and such that $X^2$ is normal. By Corollary 2.4 $X$ is not $\Delta$-paracompact. Since $X^2$ is normal, $X$ is functionally $\Delta$-normal. For a specific example of such a space we can use a subspace of the Niemytski Plane $\Gamma$, reviewed below. Assuming $\text{MA}+\neg \text{CH}$ it is well known that if $S \subseteq \mathbb{R}$ is an uncountable subset of $\mathbb{R}$ with $|S| < \omega$ then $S$ is a $Q$-set [Kun] and $X = S \times [0, \infty) \subseteq \Gamma$ (as a subspace of $\Gamma$) is a normal separable Moore space with the uncountable discrete subset $S \times \{0\}$. By [AlsPrz, Theorem 3], $X \times X$ is also normal (under $\text{MA}+\neg \text{CH}$). (In fact, $X^\omega$ is normal.)

Another corollary to Theorem 2.3 is the following statement.

Theorem 2.6. Let $X$ be $\Delta$-paracompact and pseudocompact. Then $X$ is countably compact.

Proof. Assume $X$ is not countably compact. Then there exists an infinite closed discrete subset $A \subseteq X$. By Theorem 2.3 there exists a discrete collection $\mathcal{U} = \{U_a : a \in A\}$ of open sets such that every $U_a \cap A = \{a\}$. Therefore $\mathcal{U}$ does not have a limit point contradicting pseudocompactness of $X$.

Theorem 2.6 motivates the following question.

Question 2.7. Let $X$ be $\Delta$-normal and pseudocompact. Is $X$ countably compact?

If, in Theorem 2.3, we strengthen the hypothesis to regular $\Delta$-paracompactness we can prove a much stronger conclusion using a similar argument.

Theorem 2.8. Let $X$ be regular $\Delta$-paracompact. Then $X$ is collectionwise normal.

Proof. Fix $\mathcal{F}$, a discrete family of closed subsets of $X$. Put $A = \bigcup\{F \times G : F, G \in \mathcal{F}, F \neq G\}$. Clearly, $A$ is a closed subset of $X \times X$ that misses the diagonal of $X$. Since $X$ is regular $\Delta$-paracompact there exists a locally finite open cover $\mathcal{U}$ of $X$ such that $\bigcup\{U \times \overline{U} : U \in \mathcal{U}\}$ misses $A$.

Claim 1. $\overline{U}$ can meet at most one element of $\mathcal{F}$.

For every $F \in \mathcal{F}$ put $\mathcal{U}_F$ to be the set of elements of $\mathcal{U}$ that meet $F$ and $\mathcal{V}_F$ the set of all elements of $\mathcal{U}$ that meet at least one element of $\mathcal{F} \setminus \{F\}$.

Claim 2. $\bigcup \mathcal{U}_F$ is an open neighborhood of $F$ for every $U \in \mathcal{U}$. 
Let us show that $\bigcup V_F$ does not meet $F$. For this observe first that $\bigcup V_F = \bigcup \{ V : V \in V_F \}$. This follows from local finiteness of $U$. By Claim 1 and the definition of $V_F$, we have $V$ misses $F$.

Since $X$ is normal [Buz, Proposition 2.6], there exists an open neighborhood $W_F$ of $F$ such that $W_F \subseteq \bigcup U_F$ and does not meet $\bigcup V_F$. By [ENG, Theorem 5.1.17], it suffices to show that $W_G \cap W_F = \emptyset$ for distinct $F, G \in F$. By the definitions, we have $U_G \subseteq V_F$. By our choice, $W_F$ does not meet $\bigcup V_F$ while $V_G$ is a subset of $\bigcup V_F$. \hfill \Box

Remark. In [Buz] it is shown that both paracompact spaces and GO-spaces have all the properties under consideration. It is well known that paracompact spaces and GO-spaces are collectionwise normal. Therefore Theorem 2.8 gives another unified proof of these classical facts.

Our next step in distinguishing the properties is to show that neither functional $\Delta$-normality nor $\Delta$-normality implies normality. It turns out that an example showing this is the Heath V-space $H$ [Hea], which is known to be not normal. Before we prove functional $\Delta$-normality of the Heath Space let us first describe it.

**Description of the Heath Space** $\mathbb{H}$.

Let $\mathbb{H} = E \cup U$, where $E = \mathbb{R} \times \{0\}$ and $U = \mathbb{R} \times (0, \infty)$. For $n \in \mathbb{N}$ and $x = (e, 0) \in E$ let

$$V_n(x) = \{ x \} \cup \{(s, t) \in U : (t = |s - e|) \land (0 < t < \frac{1}{n}) \}.$$  

The topology $\tau$ on $\mathbb{H}$ is induced by isolating all elements of $U$ and using the collections $\{ V_n(x) : n \in \mathbb{N} \}$ as local bases at $x \in E$. For any $x \in U, n \in \mathbb{N}$, let $V_n(x) = \{ x \}$. In any case, if $x \in X$, then $\{ V_n(x) : n \in \mathbb{N} \}$ gives a local base at $x$. The resulting space, sometimes called “Heath’s V-space”, was given by R.W. Heath in [Hea]: the space is clearly a meta-compact non-normal Moore space.

**Lemma About the Heath Space.**

**Lemma 2.9.** The Heath Space $\mathbb{H}$ is not normal [Hea] but is functionally $\Delta$-normal.

**Proof.** Use notation as given in the description above. For each $x \in \mathbb{H}$ suppose $W(x)$ is a given basic open set about $x$ and we let $W = \bigcup \{ W(x) \times W(x) : x \in \mathbb{H} \}$. Notice that $W$ is an open set in $\mathbb{H} \times \mathbb{H}$ containing the diagonal $\Delta$. The desired result will follow if we show that any such ‘canonical’ open set is actually closed in $\mathbb{H} \times \mathbb{H}$. For contradiction suppose there was some $(u, v) \in \overline{W} \setminus W$. For every $n \in \mathbb{N}$ there exists $z_n \in X$ with $(V_n(u) \times V_n(v)) \cap (W(z_n) \times W(z_n)) \neq \emptyset$. There is $x_n \in V_n(u) \cap W(z_n)$ and $y_n \in V_n(v) \cap W(z_n)$. We observe that

(*) For all $m, n \in \mathbb{N}, x_m, y_m \in W(z_m), x_n \rightarrow u$, and $y_n \rightarrow v$. Also, $W(z_n)$ is a basic neighborhood of $z_n$ so we may express $W(z_n) = V_{k_n}(z_n)$ for some $k_n$.

It is clearly not possible for both $u, v$ to be elements of $U$. The remaining possibilities will be considered – in each case, we will see that condition (*) is not possible, giving the contradiction.

**Case 1.** $u \in E$ and $v \in U$

Since $y_n \rightarrow v$ and $v$ is isolated we must have some $k \in \mathbb{N}$ such that $y_n = v \in W(z_n)$ for $n \geq k$; hence $u \notin W(z_n)$ for all $n \geq k$. Since the collection $\{ W(x) : x \in E \}$ is point-finite there must be infinite $M \subseteq \mathbb{N}$ such that $z_i = z_j$ for
all \( i, j \in M \). This would say that if \( m \in M \) then \( u \not\in W(z_m) \) but \( u \) would be a cluster point of \( \{ x_j : j \in M \} \subseteq W(z_m) \). This is a contradiction.

**Case 2.** \( u, v \in E \)

Since \( x_n \in W(z_m) \) and \( x_n \to u \) we see that the Euclidean \( ||u - z_n|| \to 0 \). Similarly \( y_n \in W(z_n) \) and \( y_n \to v \) implies that the Euclidean \( ||v - z_n|| \to 0 \). However, this is not possible since \( u \neq v \).

In any case we have a contradiction so \( W \) must be open and closed. Of course, an open and closed set is functionally open. \( \square \)

In connection with the new properties of the Heath space it may be of interest to mention that a similar (slightly more tedious) argument shows that a space given by G.M. Reed [Ree], known to be non-normal, is also functionally \( \Delta \)-normal. This space is a continuously symmetrizable Moore space which is not submetrizable. At the same time, the preceding argument breaks down at Case 1 for the Niemytski Plane, often referred to as “Tangent Disk Plane”. We show that the Niemytski Plane and a modified version are not \( \Delta \)-normal. The modified version is actually more interesting in this context.

**Description of the Niemytski Plane.** Let \( \Gamma = \mathbb{R} \times [0, \infty) \). For \( p \in \Gamma \), we define an open local base \( \{ U_n(p) \}_{n=1}^{\infty} \) at \( p \) as follows: If \( d \) is the usual Euclidean metric on \( \mathbb{R}^2 \) and \( p = (p_1, p_2) \), with \( p_2 > 0 \), let \( U_n(p) = \{ z \in \mathbb{R} \times (0, \infty) : d(p, z) < 1/n \} \). If \( p_2 = 0 \), let \( U_n(p) = \{ p \} \cup \{ z \in \Gamma : d((p_1, 1/n), z) < 1/n \} \). In this case we are describing a neighborhood consisting of \( \{ p \} \) along with an open disk tangent to the \( x \)-axis at \( p \). If \( \mathcal{G}_n = \{ U_n(p) : p \in \Gamma \} \) it is straightforward to verify that \( \{ \mathcal{G}_n \}_{n=1}^{\infty} \) is a development for the completely regular space \( \Gamma \). This space \( \Gamma \) is called the Niemytski Plane.

**Lemma About the Niemytski plane.**

**Lemma 2.10.** The Niemytski plane and a “modified” Niemytski plane are not \( \Delta \)-normal.

**Proof.** Modify the Niemytski plane by isolating the points above the \( x \)-axis and let the resulting space, with the stronger topology, be denoted by \( \Lambda \). Otherwise, use notation as given in the description above. Express \( \Lambda = E \cup U \), where \( E = \mathbb{R} \times \{ 0 \} \) and \( U = \mathbb{R} \times (0, \infty) \) and decompose \( E = \bigcup_{n \in \mathbb{N}} F_n \) such that, for all \( m \in \mathbb{N} \), \( D_m = \{ x \in E : (x, 0) \in F_m \} \) is dense in \( \mathbb{R} \).

For \( n \in \mathbb{N} \) and \( p \in F_n \) suppose \( W(p) = U_n(p) \). For \( p \in U \) let \( W(p) = \{ p \} \). Each \( W(p) \) is a basic open set about \( p \) and \( W = \bigcup \{ W(p) \times W(p) : p \in \Lambda \} \) is an open set about \( \Delta_\Lambda \) in \( \Lambda \times \Lambda \). For each \( p \in \Lambda \) let \( V(x) \) denote any basic open set about \( p \) in \( \Lambda \) with \( V(p) \subseteq W(p) \) and let \( V = \bigcup \{ V(p) \times V(p) : p \in \Lambda \} \). The desired result will hold if we show that for this pair \( V, W \) of ‘canonical’ open sets, \( \overline{V} \not\subseteq W \). Every \( V(p) \) is a basic neighborhood of \( p \) so for \( p \in E \) we may express \( V(p) = U_{k_p}(p) \) for some \( k_p \in \mathbb{N} \). By Baire Category there exists \( m \in \mathbb{N} \) and a subset \( D \subseteq E = \mathbb{R} \times \{ 0 \} \) such that \( k_p = m \), for all \( p \in D \), and \( D \) is Euclidean dense in some “open interval” \( J \subseteq E \). Pick some fixed \( q = (q_1, 0) \in F_{m+1} \cap J \) and some \( r = (r_1, r_2) \in U_m(q) \setminus U_{m+1}(q) \). Since \( U_{m+1}(q) = W(q) \) and \( r \not\in W(q) \) we see that \( (q, r) \not\in W \). We will show that \( (q, r) \in \overline{V} \). To this end, pick a strictly increasing sequence \( \{ x_n \}_{n \in \mathbb{N}} \) of real numbers, with each \( (x_n, 0) \in D \) and \( x_n \to q_1 \) (converging in \( \mathbb{R} \)). Let \( p_n = (x_n, 0) \). Notice that there exists \( k \in \mathbb{R} \) such that \( r \in U_m(p_n) \) for
all $n > k$. Also, for any neighborhood $T(q)$ of $q$ there would be some $n > k$ such that $T(q) \cap U_n(p_n) \neq \emptyset$. This says that in $\Lambda \times \Lambda$ we do have $$(q, r) \in \text{cl} \left( \bigcup U_m(p_n) \times U_m(p_n) : n > k \right) = \text{cl} \left[ \bigcup V(p_n) \times V(p_n) : n > k \right] \subseteq \bigcup \bigwedge.$$ Hence, $(q, r) \in \bigwedge \setminus W$. That completes the verification that the modified Niemytski plane $\Lambda$ is not $\Delta$-normal. Clearly, this also shows that the (usual) Niemytski plane $\Gamma$ is not $\Delta$-normal. □

K. P. Hart showed in [Har] that one classical example of a monotonically normal hereditarily countably paracompact space is not $\Delta$-normal. Our final goal in this section is to construct a normal space which is not $\Delta$-normal and has an additional nice feature, namely, a $G_{\delta}$-diagonal. To kill $\Delta$-normality we will make sure that our space does not have a regular $G_{\delta}$-diagonal. Recall that a space $X$ has a (regular) $G_{\delta}$-diagonal if the diagonal of the square is the intersection (of closures) of countably many open sets about the the diagonal. The proof of the next statement is obvious and is therefore omitted.

**Remark.** Let $X$ have a $G_{\delta}$-diagonal. If $X$ is $\Delta$-normal then $X$ has a regular $G_{\delta}$-diagonal.

**Description of Bing’s Examples G and H.**

**“Example G”** [Bin]:

Let $\kappa$ be an uncountable cardinal, let $Q$ denote the power set of $\kappa$ and $Z = 2^Q$. For every $\alpha \in \kappa$ let $e_\alpha$ denote the element of $Z$ such that, for all $A \in Q$, $e_\alpha(A) = 1 \iff \alpha \in A$ and let $E = \{e_\alpha : \alpha \in \kappa\}$.

Induce a topology on $Z$ by letting all elements of $Z \setminus E$ be isolated and let the elements of $E$ have neighborhoods inherited from the product topology on $2^Q$. It is well-known [Bin] that the resulting space $Z$ is normal but not collectionwise-Hausdorff since the elements of the closed discrete subset $E$ cannot be separated. In fact, no uncountable subset of $E$ has a separation in $Z$. (This can be shown using a Delta-system Lemma [Kun] argument on the finite subsets of $Q$ which determine the basic neighborhoods of the elements of $E$.)

For every $\alpha \in \kappa$, the basic neighborhoods of $e_\alpha$ are determined by the finite subsets $F$ of $Q$ – that is, $U(\alpha, F) = \{g \in Z : g(A) = e_\alpha(A) \}$ for all $A \in Q, g(A) = e_\alpha(A)$ gives a basic open set about $e_\alpha$. For such $\alpha, F$, we will have need to keep track of the elements of $F$ which contain $\alpha$ and those which do not contain $\alpha$ so we may always decompose $F$, relative to $\alpha$, as $F = F' \cup F''$ where $F' = \{A \in F : \alpha \in A\}$ and $F'' = \{A \in F : \alpha \not\in A\}$.

**A useful observation:**

For any two basic neighborhoods, $U(\alpha, F' \cup F''), U(\beta, E' \cup E'')$, it is true that $U(\alpha, F' \cup F'') \cap U(\beta, E' \cup E'') \neq \emptyset$ if and only if $F' \cap E'' = \emptyset$ and $F'' \cap E' = \emptyset$.

**“Example H”** [Bin]:

The topological space $X$ described below is related to the space $Z$ given above so, using the notation described above, let $X = E \cup ((Z \setminus E) \times \omega)$. The elements of $X \setminus E$ are isolated and elements of $E$ have neighborhoods as follows. For $\alpha \in \kappa$, $n \in \omega$ and $F \in [Q]^{< \omega}$ let $W(\alpha, n, F) = \{e_\alpha\} \cup (\bigcup_{k>n}(U(\alpha, F) \setminus \{e_\alpha\}) \times \{k\})$.

The sets $W(\alpha, n, F)$ give a local base about $e_\alpha$. It is well-known that the resulting space $X$, with this topology, is a normal $\sigma$-space (hence has a $G_{\delta}$-diagonal) which is
Lemma About the Bing Examples.

Lemma 2.11. (a) For any uncountable κ, Z and X are normal but not Δ-normal.

(b) If κ > ℵ then X is a normal σ-space (with a $G_δ$-diagonal) but without a regular $G_δ$-diagonal. So, X is not Δ-normal.

(c) If κ ≤ ℵ then X is a normal submetrizable space (0-dimensional weaker metric) which is not Δ-normal.

Proof. Verification of (a).

Let $G = \bigcup_{\alpha < \kappa} U(\alpha, F_\alpha) \times U(\alpha, F_\alpha)$ where, for each $\alpha \in \kappa$, $F_\alpha = \{\{\alpha\}, \kappa \setminus \{\alpha\}\}$. Now, G is an open set in $Z \times Z$ with $\Delta_Z \cap E \subseteq G$. Since the elements of $Z \setminus E$ are isolated it is enough to show there is no open $\tilde{H}$ in $Z \times Z$ with $\Delta_Z \cap E \subseteq \tilde{H} \subseteq \text{cl} \ H \subseteq G$. For this we need only consider $H$ of form $H = \bigcup_{\alpha < \kappa} U(\alpha, E_\alpha) \times U(\alpha, E_\alpha)$.

The sets $E_\alpha$, $\alpha \in \kappa$, are all finite so the Delta-system Lemma says that there exists uncountable $\Lambda \subseteq \kappa$ and some root $R$ such that $E_\alpha \cap E_\beta = R$ for all distinct $\alpha, \beta \in \Lambda$. We also may assume such $\Lambda$ such that $\mathcal{E}_{\alpha} \cap \mathcal{E}_{\beta} = R$ for all distinct $\alpha, \beta \in \Lambda$. Pick distinct $\alpha, \beta \in \Lambda$. Notice that $(e_\alpha, e_\beta) \in X \setminus G$. It will suffice to show that $(e_\alpha, e_\beta) \in \text{cl} \ H$. To this end, suppose $U(\alpha, T_\alpha) \times U(\beta, T_\beta)$ is a basic open set about $(e_\alpha, e_\beta)$. Pick some $\gamma \in \Lambda$ such that $(T'_\alpha \cup T'_\beta) \cap \mathcal{E}_\gamma \subseteq R'$ and $(T''_\alpha \cup T''_\beta) \cap \mathcal{E}_\gamma \subseteq R''$. These are compatibility conditions which, along with the choice of $\alpha, \beta \in \Lambda$, allow us to find $f \in U(\alpha, T_\alpha) \cap U(\gamma, E_\gamma)$ and $g \in U(\beta, T_\beta) \cap U(\gamma, E_\gamma)$. That is, $(f, g) \in U(\alpha, T_\alpha) \times U(\beta, T_\beta) \cap \tilde{H} \neq \emptyset$. This shows that $(e_\alpha, e_\beta) \in \text{cl} \ H \setminus G$, as desired.

That concludes the argument that Z is not Δ-normal. To see that the σ-space X is not Δ-normal we need a similar but slightly more tedious argument. In X start with $G = \bigcup_{n \in \omega} W(\alpha, 0, F_\alpha) \times W(\alpha, 0, F_\alpha)$ where each $F_\alpha = \{\{\alpha\}, \kappa \setminus \{\alpha\}\}$ as above. Now, as in the above paragraph, it can be shown that there is no open $H$ in $X \times X$ with $\Delta_X \cap E \subseteq H \subseteq \text{cl} \ H \subseteq G$.

Proof. Verification of (b).

We have the need for a technical lemma (working in Z) which will help with showing that X does not have a regular $G_δ$-diagonal.

Sublemma. Assume $\kappa > \ell$. Let $\{V_\gamma\}_{\gamma \in \omega}$ be any sequence of open collections in X, each covering $E$. Suppose, for every (unordered) $\alpha, \beta \in \kappa$, there is assigned a pair $U(\alpha, F_{\alpha\beta}(\alpha)), U(\beta, F_{\alpha\beta}(\beta))$ of basic neighborhoods of $e_\alpha, e_\beta \in E$, respectively. Then, there exists $\sigma, \gamma \in \kappa$ and for all $k \in \omega$ there exists $V_k \in V_\gamma$ such that $U(\sigma, F_{\sigma\gamma}(\sigma)) \cap V_k \neq \emptyset$ and $U(\gamma, F_{\sigma\gamma}(\gamma)) \cap V_k \neq \emptyset$.

Proof. We may assume every $V_\alpha$ is a collection (of basic neighborhoods) of the form $V_\alpha = \{U(\alpha, F_{\alpha\alpha}(\alpha)) : \alpha \in \kappa\}$. Using the decomposition of each $E_{\alpha\alpha}$ as described earlier, let $L'_\alpha = \bigcup_{n \in \omega} E'_{\alpha\alpha}, L''_\alpha = \bigcup_{\alpha, n} E''_{\alpha\alpha}$, and $L_\alpha = L'_\alpha \cup L''_\alpha$ for every $\alpha \in \kappa$.

Observe that the sets $L_\alpha$, $\alpha \in \kappa$, are all countable so the Delta-system Lemma says that there exists uncountable $\Lambda \subseteq \kappa$ and some root $R$ such that $L_\alpha \cap L_\beta = R$ for all distinct $\alpha, \beta \in \Lambda$. We also may assume such $\Lambda$ such that $L'_\alpha \cap L'_\beta = \emptyset$ for all $\alpha, \beta \in \Lambda$ (see [Bur, Theorem 1.2]).
Let $\sigma, \gamma \in \Lambda$. Since no element of the finite set $(F_{\sigma \gamma}(\sigma) \cup F_{\sigma \gamma}(\gamma)) \setminus \mathcal{R}$ can be an element of more than one $\mathcal{L}_\alpha$, $\alpha \in \Lambda$, there must be some $p \in \Lambda$ such that
\[
\mathcal{L}_p \cap ((F_{\sigma \gamma}(\sigma) \cup F_{\sigma \gamma}(\gamma)) \setminus \mathcal{R}) = \emptyset.
\]

Let $\mathcal{R}' = \mathcal{R} \cap \mathcal{L}_p'$ and let $\mathcal{R}'' = \mathcal{R} \cap \mathcal{L}_p''$. By the conditions on $\mathcal{L}_\alpha$, $\alpha \in \Lambda$, we see that $\mathcal{R}' = \mathcal{R} \cap \mathcal{L}_\alpha'$ and $\mathcal{R}'' = \mathcal{R} \cap \mathcal{L}_\alpha''$ for all $\alpha \in \Lambda$. In particular, if $A \in \mathcal{R}'$ then $p, \sigma, \gamma \in A$ and if $B \in \mathcal{R}''$ then $p, \sigma, \gamma \notin B$. These conditions tell us that $e_p|\mathcal{L}_p$ and $e_p|F_{\sigma \gamma}(\sigma)$ have a common extension $g$. Also, it is possible to find a common extension $h$ of $e_p|\mathcal{L}_p$ and $e_p|F_{\sigma \gamma}(\gamma)$. Now, for all $n \in \omega$ we have $g \in U(\sigma, F_{\sigma \gamma}(\sigma)) \cap U(p, E_{n,p}) \neq \emptyset$ and $h \in U(\gamma, F_{\sigma \gamma}(\gamma)) \cap U(p, E_{n,p}) \neq \emptyset$, as desired. That completes the proof of the sublemma. \hfill \Box

The Sublemma provides a crucial property of $Z$ which will tell us that the another related example $X$ by Bing (a normal $\sigma$-space) does not have a regular $G_{\delta}$-diagonal (when $\kappa > \varsigma$).

We now formalize the statement of Lemma 2.11(b) in the form of an example and finish the proof.

**Example.** When $\kappa > \varsigma$ the space $X$ above is a normal $\sigma$-space (with a $G_{\delta}$-diagonal) but $X$ does not have a regular $G_{\delta}$-diagonal.

**Proof.** The verification that $X$ does not have a regular $G_{\delta}$-diagonal will follow from Sublemma given above. This uses Zenor’s characterization of spaces with a regular $G_{\delta}$-diagonal [Zen]. The statement of the Sublemma can easily be upgraded to a statement about $X$ in the following way:

Assume $\kappa > \varsigma$. Let $\{V_n\}_{n \in \omega}$ be any sequence of open collections in $X$, each covering $E$. Suppose, for every (unordered) $\alpha, \beta \in \kappa$, there is assigned a pair $W(\alpha, n_{\alpha \beta}, F_{\alpha \beta}(\alpha)), W(\beta, n_{\alpha \beta}, F_{\alpha \beta}(\beta))$ of basic neighborhoods of $e_{\alpha}, e_{\beta} \in E$, respectively. Then, there exists $\sigma, \gamma \in \kappa$ and for all $k \in \omega$ there exists $V_k \in V_k$ such that
\[
W(\sigma, n_{\alpha \gamma}, F_{\sigma \gamma}(\sigma)) \cap V_k \neq \emptyset \text{ and } W(\gamma, n_{\alpha \gamma}, F_{\sigma \gamma}(\gamma)) \cap V_k \neq \emptyset.
\]

Combining the above paragraph with the following characterization by Zenor shows that $X$ does not have a regular $G_{\delta}$-diagonal.

**Zenor’s Characterization.** [Zen] A space $X$ has a regular $G_{\delta}$-diagonal if and only if there exists a sequence $\{V_n\}_{n \in \omega}$ of open covers of $X$ such that for any distinct $x, y \in X$ there exist open $U_x, U_y$ about $x, y$ respectively and $k \in \omega$ such that, for all $V \in V_k$, either $U_x \cap V = \emptyset$ or $U_y \cap V = \emptyset$.

That concludes the verification of (b). \hfill \Box

**Proof.** Verification of (c).

Assume $\kappa \leq \varsigma$. Although not really necessary it may be worthwhile to point out that if $X$ does have a weaker metric topology then this weaker topology, restricted to $E$, must actually be separable. Assume $\tau'$ gives a weaker topology on $X$ with metric $d$ and let $\tau$ denote the original topology on $X$. Observe that $E$, with this relative metric topology, must be separable since it cannot contain any uncountable discrete sets: If there was an uncountable discrete set $B \subseteq E$ then there would exist an uncountable closed (in metrizable $(X, \tau')$ subset $A \subseteq B$. Since $(X, \tau')$ is paracompact there would exist an open collectionwise separation of $A$ by a pairwise disjoint collection of open sets $\{W_a : a \in A\}$. However, this separation by $\{W_a : a \in A\}$ would also be a separation of the uncountable set $A$ in $(X, \tau)$ (original topology). This is not possible by an earlier remark.
Now that we know the set $E$ must be separable with any possible weaker metric topology, this suggests that we build the topology starting with $E$ "identified as a subspace" $\tilde{E}$ of the irrationals $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$. Assuming $|E| = \kappa \leq \mathfrak{c}$ there is a countable collection $\{J_n : n \in \omega\}$ of subsets of $E$, such that the corresponding subsets of $\tilde{E}$ would be a clopen base for $\tilde{E}$ as a subspace of $\mathbb{P}$. For convenience assume the collection $\{J_n : n \in \omega\}$ is closed under finite intersections and that for any $J_n$ the set $E \setminus J_n = J_k$, for some $k$. (That is, the elements of $\{J_n : n \in \omega\}$ come in complementary pairs.) We need to properly expand each of these sets $J_n$ into $X$ in order to end up with a metrizable topology. First, in $Z$, for every complementary pair $J_n, J_k$, use normality in order to find open sets $V_n, V_k$ such that $J_n \subseteq V_n, J_k \subseteq V_k$ and $V_n \cap V_k = \emptyset$. Re-enumerate, if necessary, and we may assume that the collection $\{V_n : n \in \omega\}$ forms a $\sigma$-locally finite base for a regular topology on $X$. Hence this is a weaker $\Delta$-normality imply functional $\Delta$-normality?

3. Properties in Continuous Images

In this section we begin to study the behavior of the properties under continuous maps. In [Har, Example 3.3], KP Hart gave an example of a perfect map which does not preserve $\Delta$-normality. In fact, he observes that the diagonal $\Delta_Y$ of the domain space $Y$ has a clopen neighborhood base in $Y \times Y$ (and hence $Y$ is actually functionally $\Delta$-normal). We will show that a perfect image of a $\Delta$-paracompact space need not be $\Delta$-paracompact and that none of the $\Delta$-separation properties under consideration is preserved by open maps. We start with the following obvious statement.

**Lemma 3.1.** Let $X_i$, for all $i \in I$, share a $\Delta$-separation property. Then $\oplus_{i \in I} X_i$ has the same property.

**Proof.** The conclusion follows from the fact that $\Delta_{\oplus_{i \in I} X_i} = \oplus_{i \in I} \Delta_{X_i}$. \hfill $\square$
To demonstrate the discussed failures we need to describe one space first that has no $\Delta$-separation properties. Then we will show that it is an open or a closed image of spaces with some $\Delta$-separation properties.

**Example 3.2.** Let $Z = \{ (\alpha, \beta) \in \omega_1 \times (\omega_1 + 1) : \alpha \leq \beta \}$ and let $p$ be the partition on $Z \times \{ 0, 1 \}$ whose only non-trivial elements are in forms

1. $\{ (\alpha, \alpha, 0), (\alpha, \omega_1, 1) \}$;
2. $\{ (\alpha, \omega_1, 0), (\alpha, \alpha, 1) \}$.

Then $Y = p(Z \times \{ 0, 1 \})$ has no $\Delta$-separation properties.

*Proof.* It is shown in [Buz, Example 2.12] that $Z$ is $\Delta$-paracompact. In [Buz, Corollary 2.7] it is shown that $\Delta$-normality together with $\Delta$-paracompactness imply normality. Thus, since $Z$ is $\Delta$-paracompact and not normal it is not $\Delta$-normal.

If a space $X$ is functionally $\Delta$-paracompact, regular $\Delta$-paracompact, or functionally $\Delta$-normal then it is $\Delta$-paracompact or $\Delta$-normal. Therefore, we need to show that $Y$ is neither $\Delta$-normal nor $\Delta$-paracompact.

One can think of $Z$ as a transfinite right triangle with the transfinite acute vertex deleted. That is, the non-compact legs of triangles $Z \times \{ 0 \}$ and $Z \times \{ 1 \}$ are glued up with the hypotenuses of $Z \times \{ 1 \}$ and $Z \times \{ 0 \}$, respectively. Since $Y$ contains a closed copy of $Z$ and $Z$ is not $\Delta$-normal, $Y$ is not $\Delta$-normal either. To show that $Y$ is not $\Delta$-paracompact, consider the map $f$ defined as follows:

1. $f(\{ (\alpha, \alpha, 0), (\alpha, \omega_1, 1) \}) = \{ (\alpha, \alpha, 1), (\alpha, \omega_1, 0) \}$;
2. $f(\{ (\alpha, \alpha, 1), (\alpha, \omega_1, 0) \}) = \{ (\alpha, \alpha, 0), (\alpha, \omega_1, 1) \}$;
3. $f(\{ (\alpha, \beta, 0) \}) = \{ (\alpha, \beta, 1) \}$, where $\alpha < \beta < \omega_1$;
4. $f(\{ (\alpha, \beta, 1) \}) = \{ (\alpha, \beta, 0) \}$, where $\alpha < \beta < \omega_1$.

Geometrically, $f$ is the rotation that maps the triangle $p(Z \times \{ 0 \})$ onto the triangle $p(Z \times \{ 1 \})$ and vice versa in the most natural manner. Thus, $f$ is continuous. Therefore $A = \{ (y, f(y)) : y \in Y \}$ is a closed subset of $Y \times Y$. Since $f$ is fixed point free, $A$ does not meet $\Delta_Y$. It is left to show now that $\Delta_Y$ cannot be separated from $A$ by a locally finite cover consisting of open squares. For this fix an arbitrary locally finite open cover $U$ of $Y$. Since $Y$ is countably compact, $U$ is finite. For our further discussion we need the following claim which follows from the Pressing Down Lemma.

**Claim (folklore).** Let $V$ be an open neighborhood of $\{ (\alpha, \alpha) : \alpha < \omega_1 \}$ and $W$ an open neighborhood of $\{ (\alpha, \omega_1) : \alpha < \omega_1 \}$ in $\omega_1 \times (\omega_1 + 1)$. Then $V$ meets $W$.

Since $U$ is finite there exists $U \in U$ that contains almost all of $p(\{ (\alpha, \alpha, 0) : \alpha < \omega_1 \})$. More precisely, there is $\gamma < \omega_1$ such that $\{ (\alpha, \alpha, 0), (\alpha, \omega_1, 1) \} \in U$ for every countable $\alpha > \gamma$.

If one looks at the trace of $U$ on $p(Z \times \{ 0 \})$ one sees a neighborhood of an uncountable tail of the hypotenuse in the triangle. If one looks at the trace of $U$ on $p(Z \times \{ 1 \})$ one sees a neighborhood of an uncountable tail of the non-compact leg. Therefore, by Claim, there exist $\alpha, \beta$ with $\alpha < \beta$ such that $\langle \alpha, \beta, 0 \rangle$ and $\langle \alpha, \beta, 1 \rangle \in U$. But $f(\langle \alpha, \beta, 0 \rangle) = \langle \alpha, \beta, 1 \rangle$. Therefore, $\langle \alpha, \beta, 0 \rangle$ and $\langle \alpha, \beta, 1 \rangle$ is in $A \cap (U \times U)$. The proof of non $\Delta$-paracompactness of $Y$ is complete. \hfill $\square$

By [Buz, Example 2.12], $Z$ is $\Delta$-paracompact. By Lemma 3.1, $Z \times \{ 0, 1 \}$ is $\Delta$-paracompact. Clearly, the quotient map $p$ is perfect. Thus we have the following.
Corollary 3.3. Δ-paracompactness is not preserved by quotient maps or perfect maps.

Proposition 3.4. None of the Δ-separation properties are preserved by open maps.

Proof. As before, let $Z = \{ (\alpha, \beta) \in \omega_1 \times (\omega_1 + 1) : \alpha \leq \beta \}$ and $Y = p(Z \times \{0, 1\})$. For every $\gamma < \omega_1$, put $Z_\gamma = \{ (\alpha, \beta) \in Z : \alpha \leq \gamma \}$ and $Y_\gamma = p(Z_\gamma \times \{0, 1\})$. The space $Y_\gamma$ is a closed subspace of a compactum. Therefore, $Y_\gamma$ has all Δ-separation properties. Let $i_\gamma$ be the natural embedding of $Y_\gamma$ into $Y$. Define $i : \oplus_{\gamma \in \omega_1} Y_\gamma \to Y$ by letting $i(x) = i_\gamma(x)$ if $x \in Y_\gamma$. Since $Y_\gamma$ is open in $\oplus_{\gamma \in \omega_1} Y_\gamma$ and in $Y$, the map $i$ is open. By Lemma 3.1, $\oplus_{\gamma \in \omega_1} Y_\gamma$ has all Δ-separation properties. By Example 3.2, $i(\oplus_{\gamma \in \omega_1} Y_\gamma)$ has none. □

It is clear however that if $X$ is countably compact and Δ-paracompact then any open image of $X$ is Δ-paracompact as well. There is still a hope that some of the Δ-separation properties are preserved by perfect or perfect open or closed maps. An encouraging observation in this direction was made in [Har], namely, that Δ-normality is preserved by perfect open maps.

Question 3.5. Let $X$ have a Δ-separation property and suppose $f : X \to Y$ is continuous. Which conditions on $f$ and/or $X$ and/or $Y$ guarantee that $Y$ has the same property?

References


