Abstract: In our main result we prove that a continuous function \( f : \mathbb{R} \to \mathbb{R} \) fixes a point iff \( \tilde{f} : \beta\mathbb{R} \to \beta\mathbb{R} \) does, where \( \tilde{f} \) is the continuous extension of \( f \) over the Čech-Stone compactification. In the second half of the paper the statement of this theorem with \( \mathbb{R} \) replaced by an arbitrary \( X \) is taken for a definition of a property and a number of results about the property are proved.

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1. Introduction

In [KAT], it is shown that a function \( f \) from a discrete space to itself fixes a point iff \( \tilde{f} \) does, where \( \tilde{f} \) is the continuous extension over the Čech-Stone compactification. This theorem leads naturally to the following general problem.

Problem. Let \( f : X \to X \) be a continuous function such that \( \tilde{f} : \beta X \to \beta X \) fixes a point. What conditions on \( X \) and/or \( f \) guarantee that \( f \) fixes a point too?

In [DOU], it is shown that even metrizability does not guarantee the desired conclusion. Namely, E. van Douwen showed that there exists a fixed-point free homeomorphism of \( \oplus_n S^n \) onto itself such that \( f \) fixes a point, where \( S^n \) is the \( n \)-dim sphere. In the same paper, E. van Douwen showed, in particular, that if \( X \) is a paracompact space of finite dimension and \( f \) is a homeomorphism of \( X \) into \( X \) then \( f \) fixes a point iff \( \tilde{f} \) does. This work of van Douwen inspired many papers by other authors. A good account of the results related to the above problem as well as historical comments are given in [MI2]. Many known theorems related to this topic are restricted to special classes of continuous functions. It is known that van Douwen’s theorem cannot be extended to the class of all continuous functions. Mazur constructed a continuous finite-to-one map \( f \) of the space of irrationals into itself such that \( f \) is fixed-point free and \( \tilde{f} \) is not. This example was first stated in [K&S] (see also [MI2] for a detailed proof).

In this paper we strengthen van Douwen’s statement for the space of real numbers. Namely, we show (Theorem 2.4) that a continuous map \( f \) from \( R \) to \( R \) fixes a point iff \( \tilde{f} \) does. Other results are obtained.

In notation and terminology we will follow [ENG]. All spaces are assumed to be Tychonoff. If \( f : X \to Y \) is a continuous map, the continuous extension of \( f \) over the Čech-Stone compactifications is denoted by \( \tilde{f} \).

2. Fixed Point Maps on the Reals
Lemma 2.1. Let $f: (−∞, 0) \to R$ be continuous and $f(x) > x$ for all $x \in (−∞, 0]$. Then there exist closed $F_1, F_2, F_3, F_4 \subset (−∞, 0]$ such that

1. $(-∞,0] = F_1 \cup \ldots \cup F_4$; and
2. $f(F_i) \cap F_i = \emptyset$ for $i = 1, \ldots, 4$.

Proof. Note that for each $x \in (−∞, 0]$ there exists an open $U_x$ in $(-∞,0]$ such that $f(U_x)$ is strictly to the right of the max $U_x$. This follows from the continuity of $f$ and the inequality $f(x) > x$ for all $x$. Let $I$ be the set of all closed segments $I$ of $(−∞, 0]$ such that $f(I)$ is strictly to the right of the max $I$. Since $(−∞, 0]$ is ordered, Lindelöf, and has the maximum there exists $\{I_n\}_{n\in\omega} \subset I$ such that

A1: $(-∞,0] = \bigcup_{n\in\omega} I_n$;
A2: If $i \neq j$ then $I_i$ and $I_j$ share at most one point;
A3: If $I_i \cap I_j \neq \emptyset$ then $|i - j| \leq 1$.

That is, we assume that $I_0, \ldots, I_n, \ldots$ appear in $(−∞, 0]$ in decreasing order with $I_0$ being the rightmost and $I_{n+1}$ being next to $I_n$ on the left.

Step 0: Pick any $a, b > 0$ such that $f(I_0) \subset (a, b)$. Such $a$ and $b$ exist because $f(I_0)$ is compact and strictly to the right of 0. Put $J_0^1 = [\min I_0, a]$, $J_0^2 = [a, b]$, $J_0^3 = [b, b + 1]$, and $J_0^4 = [b + 1, \infty)$.

Assumption: Assume that for every $k < m$ we have defined closed sets (not necessarily segments) $J_k^1, \ldots, J_k^4$ with the following properties:

B1: $[\min I_k, \infty) = J_k^1 \cup \ldots \cup J_k^4$;
B2: $f(J_k^i) \cap J_k^i = \emptyset$ for $i = 1, \ldots, 4$;
B3: If $J_k^i \cap J_k^j \neq \emptyset$ and $i \neq j$ then $\{i, j\}$ is neither $\{1, 3\}$ nor $\{2, 4\}$.

Note that B1 and B3 trivially hold for $k = 0$. To verify B2, recall that $f(J_0^1) = f(I_0) \subset (a, b)$ and $(a, b) \cap J_0^j = \emptyset$. For $i = 2, 3, 4$, we have $f(J_0^i) = \emptyset$ since $J_0^i$ does not meet the domain of $f$, which is $(−∞, 0]$. Hence B2 holds.

Step m: Put

$$J_m^1 = [f^{-1}(J_{m-1}^4) \cap I_m] \cup J_{m-1}^3;$$
$$J_m^2 = [f^{-1}(J_{m-1}^3) \cap I_m] \cup J_{m-1}^4;$$
$$J_m^3 = [f^{-1}(J_{m-1}^1) \cap I_m] \cup J_{m-1}^1;$$
$$J_m^4 = [f^{-1}(J_{m-1}^2) \cap I_m] \cup J_{m-1}^2.$$

It is clear that $J_m^i$ is closed for each $i = 1, \ldots, 4$. Let us check B1-B3. To show B1, first recall that $I_{m-1}$ is next to $I_m$ on the right. Therefore, $[\min I_m, \infty) = I_m \cup [\min I_{m-1}, \infty)$. The set $[\min I_{m-1}, \infty)$ is $J_{m-1}^1 \cup \ldots \cup J_{m-1}^4$ because of the second summands in the definitions of $J_m^i$’s. The set $I_m$ equals $f^{-1}([\min I_{m-1}, \infty)) \cap I_m$. This is because $f(I_m)$ is strictly to the right of $I_m$ and all points greater than
the right endpoint of $I_m$ are in $[\min I_{m-1}, \infty)$. Therefore, $I_m$ is in $J_m^1 \cup \ldots \cup J_m^4$ because of the first summands in the definitions of $J_m^i$’s.

Let us demonstrate that B2 holds, for example, for $J_m^1$. For this, we need to show that the following four statements are true:

a. $f(f^{-1}(J_{m-1}^1) \cap I_m)$ does not meet $f^{-1}(J_{m-1}^3) \cap I_m$;

b. $f(f^{-1}(J_{m-1}^1) \cap I_m)$ does not meet $J_{m-1}^3$;

c. $f(J_{m-1}^3)$ does not meet $f^{-1}(J_{m-1}^1) \cap I_m$;

d. $f(J_{m-1}^3)$ does not meet $J_{m-1}^3$.

Statement (a) follows from the fact that $f(I_m)$ is strictly to the right of $\max I_m$. To show (b), observe that the first set is a subset of $J_{m-1}^3$ and the second is that of $J_{m-1}^3$. Now apply the inductive assumption B3 for $m-1$. To show (c), observe that $J_{m-1}^3$ is in $[\max I_m, \infty)$. Therefore, $f(J_{m-1}^3)$ is strictly to the right of $I_m$. Finally, (d) follows from the inductive assumption B2 for $m-1$.

Let us demonstrate that B3 holds, for example, for $\{1, 3\}$. We need to prove the following four statements.

A. $[f^{-1}(J_{m-1}^1) \cap I_m]$ does not meet $[f^{-1}(J_{m-1}^3) \cap I_m]$;

B. $[f^{-1}(J_{m-1}^1) \cap I_m]$ does not meet $J_{m-1}^3$;

C. $J_{m-1}^3$ does not meet $[f^{-1}(J_{m-1}^3) \cap I_m]$;

D. $J_{m-1}^3$ does not meet $J_{m-1}^3$.

To show (A), use the inductive assumption that $J_{m-1}^1 \cap J_{m-1}^3 = \emptyset$ and the fact that $f$ is a function. To show (B), assume the contrary and pick $p$ in the intersection. Then $f(p) \in f(f^{-1}(J_{m-1}^3)) = J_{m-1}^3$ and $f(p) \in f(J_{m-1}^3)$, contradicting B2 for $m-1$. Statement (C) is analogous to (B). Statement (D) follows from B3 for $m-1$.

The inductive construction is complete. Put $F_i = \bigcup \{J_m^i : m \text{ is odd}\} \cap (-\infty,0]$. Let us show that $F_i$’s are as desired.

To show that $(-\infty,0] = F_1 \cup \ldots \cup F_4$, pick $p \in (-\infty,0]$. Let $p \in I_m$. By B1, $p \in J_m^i$ for some $i \in \{1, ..., 4\}$. Assume, for example, $p \in J_m^2$. If $m$ is odd, then $p \in F_2$ by the definition of $F_2$. If $m$ is even, then $J_m^2 \subset J_m^{m+1}$ by construction. Since $m+1$ is odd we have $p \in F_4$.

Let us show that $f(F_1) \cap F_1 = \emptyset$. Let us demonstrate it for $i = 1$. By our inductive construction, $J_{2m+3}^1$ contains $J_{2m+2}^2$ and the latter contains $J_{2m+1}^1$. Thus, $J_m^1$ contains $J_k^i$ for every odd numbers $m$ and $k$ with $m > k$. Now assume $f(F_1) \cap F_1 \neq \emptyset$. Then there exist $p \in F_1$ and odd $n$ and $m$ such that $p \in J_n^1$ and $f(p) \in J_m^1$. Pick any odd $k > \max\{n,m\}$. By our initial observation, $J_n^1, J_m^1 \subset J_k^1$. Therefore, $f(J_k^1)$ meets $J_k^1$, contradicting B2. Thus, the intersection in question is empty.

To complete the proof, we need to show that $F_i$ is closed. Since $\{I_n\}_n$ is a locally finite cover of $(-\infty,0]$ it suffices to show that $F_i \cap I_k$ is compact. Let us
demonstrate it for \( i = 1 \). Pick any \( m > k + 1 \). Since \( |m - k| > 1 \) we conclude that \( I_k \) meets neither \( I_{0+2} \) nor \( I_{m+1} \). Therefore, \( J_{m+2}^1 \cap I_k = J_{m+1}^3 \cap I_k = J_m^1 \cap I_k \).

Pick any odd \( m \) greater than \( k + 1 \). We have

\[
F_1 \cap I_k = \bigcup\{J_n^1 \cap I_k : n \text{ is odd}\} = \bigcup\{J_n^1 \cap I_k : n \text{ is odd and } n < m\} \cup \{J_m^1 \cap I_k\}
\]

One of induction requirements is that each \( J_n^1 \) is closed. Therefore, the rightmost part of the above equality is compact. The proof is complete.

\[\square\]

Lemma 2.2. Let \( f : (\infty, 0] \rightarrow R \) be continuous and \( f(x) > x \) for all \( x \in (-\infty, 0] \). Then \( f : (\infty, 0] \rightarrow \beta R \) is fixed-point free.

Proof. For each \( i = 1, \ldots, 4 \), let \( G_i = f^{-1}(F_i) \), where \( F_i \) is as in Lemma 2.1. Observe that \( f(G_i) \cap G_i = \emptyset \). Indeed, if there exists \( x \) in the intersection then \( f(x) \in f(f(G_i)) = f(F_i) \) and \( f(x) \in f(G_i) = F_i \), contradicting 2 of Lemma 2.1. Thus, \( f(G_i) \) and \( G_i \) are disjoint closed sets. Fix any \( p \in \beta(\infty, 0] \). If \( p \) is a limit point of \( G_i \) then \( \tilde{f}(p) \) is a limit point of \( f(G_i) \). Since \( R \) is normal and the sets \( G_i \) and \( f(G_i) \) are closed and disjoint, we conclude that \( \tilde{f}(p) \neq p \). Now suppose \( p \) is not a limit point of \( G_1 \cup \ldots \cup G_4 \). By 1 of Lemma 2.1, \( p \) is a limit point of \( f^{-1}(0, \infty) \). We have \( p \in \beta(\infty, 0] \) and \( \tilde{f}(p) \in \beta[0, \infty) \). Since \( f(0) > 0 \), we conclude that \( \tilde{f}(p) \neq p \).

\[\square\]

Lemma 2.3. Let \( g : [0, 1] \rightarrow R \) be continuous and \( g(x) > x \) for all \( x \in [0, 1] \). Then there exist closed \( G_1, G_2, G_3, G_4 \subset [0, 1] \) such that

1. \( [0, 1] = G_1 \cup \ldots \cup G_4 \); and
2. \( f(G_i) \cap G_i = \emptyset \) for \( i = 1, \ldots, 4 \).

Proof. Define \( f : (\infty, 1] \rightarrow R \) by letting \( f(x) = g(0) \) for every \( x \leq 0 \) and \( f(x) = g(x) \), otherwise. Now apply Lemma 2.1 for \( f \) and \((\infty, 1]\). Put \( G_i = [0, 1] \cap F_i \), where \( F_i \)'s are as in the conclusion of Lemma 2.1.

\[\square\]

Theorem 2.4. Let \( f : R \rightarrow R \) be continuous. If \( \tilde{f} \) fixes a point then so does \( f \).

Proof. Assume \( f \) does not fix a point. We need to show that \( \tilde{f} \) does not fix a point either. Since \( f \) does not fix a point, the graph of \( f \) in \( R \times R \) lies always above or always below the diagonal. If the graph is below the diagonal then reversing the order of \( R \) brings us to the case when the corresponding graph is above the diagonal. Therefore, we may assume that \( f(x) > x \) for all \( x \in R \). By Lemma 2.2, \( \tilde{f} \) does not fix any points in \( \beta(\infty, 0] \). Let us show that \( \tilde{f} \) does not fix any point in \( \beta[0, \infty) \). Let \( I_0 = [b_0, b_1] \) be any non-trivial interval and \( I_1 = [b_0, b_1] \), where \( b_1 \) is the maximum \( b \) such that \( f(b) \). Since \( f(x) > x \) for all \( x \), we have \( f(I_0) \subset I_0 \cup I_1 \). Continuing in this manner we can define \( \{I_n\}_n \) a family of closed intervals with the following properties:

4
A1: \( f(I_n) \subset I_n \cup I_{n+1}; \)

A2: \( \max f(I_n) < \max I_{n+1}; \)

A3: \( \max I_n = \min I_{n+1}; \)

A4: \( [0, \infty) = \bigcup_n I_n \) (to ensure this, take intervals of length at least 1).

By Lemma 2.3, for each \( n \) we can find \( \{I_{n,i} : i = 1, ..., 4\} \) such that

B1: \( I_{n,i} \) is compact;

B2: \( I_n = I_{n,1} \cup ... \cup I_{n,4}; \)

B3: \( f(I_{n,i}) \cap I_{n,i} = \emptyset. \)

Now for each \( i \in \{1, 2, 3, 4\} \) put \( E_i = \bigcup_{n \in \omega} I_{2n,i} \) and \( O_i = \bigcup_{n \in \omega} I_{2n+1,i}. \) These sets have the following properties:

C1: \( E_i, O_i \) are closed in \( R. \)

This follows from the fact that \( \{I_n\}_n \) is a locally finite family, which is guaranteed by A3 and A4 and the fact that each \( I_{n,i} \) is compact.

C2: \( f(E_i) \cap E_i = \emptyset, f(O_i) \cap O_i = \emptyset. \)

To see this, observe that \( f(E_i) \) is closed in \( R. \) Indeed, by A1, \( f(I_{2n,i}) \subset I_{2n} \cup I_{2n+1}. \) By A3 and A4, \( \{f(I_{2n,i})\}_n \) is a locally finite family. Therefore, \( \bigcup_n f(I_{2n,i}) = \bigcup_n f(I_{2n,i}). \) By B1 and continuity of \( f, \) the set \( f(I_{2n,i}) \) is closed in \( R. \) Now to prove \( f(E_i) \cap E_i = \emptyset \) we need to show that \( f(I_{2n,i}) \cap I_{2k,i} = \emptyset \) for any \( n, k \in \omega. \) If \( n \neq k \) apply A1 and A2. If \( n = k \) apply B3.

C3: \( [0, \infty) = E_1 \cup ... \cup E_4 \cup O_1 \cup ... \cup O_4. \)

This follows from A4 and B2.

Fix any \( p \in \beta[0, \infty). \) By C3, we may assume that \( p \) is a limit point of \( E_1. \) By C1 and C2, \( p \) is not a limit point of \( f(E_1). \) Therefore, \( f \) does not fix any point in \( \beta[0, \infty). \)

We would like to re-state Theorem 2.4 using the classical notion of coloring. A continuous map \( f : X \to X \) is said to be colorable with \( n \) colors, where \( n \in \omega \) if there exists an \( n \)-sized closed cover \( F \) of \( X \) such that \( f(F) \cap F = \emptyset \) for every \( F \in F. \) It is shown in [AFR] that each fixed-point free homeomorphism of a metrizable space \( X \) with \( \dim X \leq n \) is colorable with \( n+3 \) colors. The argument of Theorem 2.4 shows that any continuous map of \( R \) into \( R \) is finitely colorable. In [MI1] the following theorem is proved: Let \( X \) be normal and let \( f : X \to X \) be a fixed-point free continuous map. If \( f \) is finitely colorable and \( \dim X \leq n \) then \( f \) can be colored with \( n+3 \) colors. Therefore, Theorem 2.4 can be re-stated as follows.
Theorem 2.5. Any fixed-point free continuous map from $\mathbb{R}$ to $\mathbb{R}$ can be colored with 4 colors.

We finish the section with the following technical generalization of Theorem 2.4.

Theorem 2.6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $\tilde{f}(p) = p$. Then $p \in \text{Cl}_{\beta\mathbb{R}}\{x \in \mathbb{R} : f(x) = x\}$.

Proof. Put $F = \{x \in \mathbb{R} : f(x) = x\}$. We may assume that $p \in \beta\mathbb{R} \setminus \mathbb{R}$. Assume $p \notin \text{Cl}_{\beta\mathbb{R}}F$. Then there exists a discrete family $\mathcal{I}$ of closed segments such that

1. $p$ is a limit point of $\bigcup \mathcal{I}$;
2. $F \subset \mathbb{R} \setminus \bigcup \mathcal{I}$.

For each $I \in \mathcal{I}$, either $f|_{I}(x) > x$ for all $x \in I$ or $f|_{I}(x) < x$ for all $x \in I$. This follows from the fact that $f$ does not fix any point in $I$ and $I$ is connected. Therefore, we may assume that

3. $f|_{I}(x) > x$ for each $I \in \mathcal{I}$ and each $x \in I$.

Let $\mathcal{K}$ be the family of maximal connected components of $\mathbb{R} \setminus \bigcup \mathcal{I}$. We have

4. $\bigcup \mathcal{K} = \mathbb{R} \setminus \bigcup \mathcal{I}$.

Since $\mathcal{I}$ is a discrete family of closed sets, each $K \in \mathcal{K}$ is an open convex set with endpoints in $\bigcup \mathcal{I}$. By 3, if $x$ is an endpoint of $K$ then $f(x) > x$. Therefore, there exists a continuous $f_{K} : K \rightarrow \mathbb{R}$ such that

5. $f_{K}(x) = f(x)$ if $x$ is an endpoint of $K$;
6. $f_{K}$ does not fix any point.

Define $g : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$
g(x) = \begin{cases} 
  f(x) & \text{if } x \in \bigcup \mathcal{I} \\
  f_{K}(x) & \text{if } x \in K
\end{cases}
$$

By 4, $g$ is defined for all $x \in \mathbb{R}$. By 5, $g$ is a well-defined continuous function. By 6, $g$ does not fix any points in $\bigcup \mathcal{K}$. By 3, $g$ does not fix any points in $\bigcup \mathcal{I}$.

By the definition, $g$ coincides with $f$ on $\bigcup \mathcal{I}$. Since $p \in \text{Cl}_{\beta\mathbb{R}}(\bigcup \mathcal{I})$ and $\tilde{f}(p) = p$, we conclude that $\tilde{g}(p) = p$. Thus, $g$ is a fixed-point free map while $\tilde{g}$ is not, contradicting Theorem 2.4.

\[\square\]

3. Fixed Filter-to-Point Property

In this section we will show that an analogue of Theorem 2.4 holds for certain countably compact spaces. To shorten the statements of the results we introduce the following definition.
Definition 3.1. A space $X$ has the fixed filter-to-point property if every continuous map $f : X \to X$ has a fixed point iff $\tilde{f} : \beta X \to \beta X$ has a fixed point.

We will investigate this property under basic topological operations. In our first result of the section, Proposition 3.4, we describe a class of spaces that have the property in question. For this purpose we need the following two technical lemmas.

Lemma 3.2. (folklore) Let $\tau$ be an uncountable regular cardinal. If $f$ is a perfect map of $\tau$ then there exists a closed unbounded $T \subset \tau$ such that $f|_T$ is a homeomorphism.

Proof. Define $S \subset \tau$ as follows: $\alpha \in S$ iff there exists $\beta \alpha < \alpha$ such that $f(\alpha) = f(\beta \alpha)$. The correspondence $\alpha \to \beta \alpha$ defines a regressive function. Assume $S$ is stationary. By the Pressing Down Lemma (see, for example, [KUN]), there exists a stationary $S' \subset S$ and $\lambda < \tau$ such that $f(\alpha) = f(\lambda)$ for all $\alpha \in S'$. Then $f(\tau) = f(\lambda)$, contradicting the fact that $f$ is perfect.

Thus, $S$ is not stationary. Therefore, there exists a closed unbounded subset $T$ of $\tau$ that misses $S$. Clearly, $T$ is as desired.

Lemma 3.3. (folklore) Let $\tau$ be an uncountable regular cardinal. Let $T$ be an unbounded closed subset of $\tau$ and $f : T \to \tau$ a continuous map unbounded from above. Then $f(\lambda) = \lambda$ for some $\lambda \in T$.

Proof. Since $f$ is unbounded from above we can select a strictly increasing sequence $\{\alpha_n\}_n$ of elements of $T$ such that

1. $f(\alpha_{n+1}) > f(\alpha_n)$;
2. $\alpha_{n+1} > f(\alpha_n)$.

Put $\lambda = \lim_{n\to\infty} \alpha_n$. Since $T$ is closed in $\tau$ and $\tau$ is countably compact, $\lambda \in T$. Clearly, $f(\lambda) = \lambda$.

Proposition 3.4. Let $X$ be a countably compact space such that for any $z \in \beta X \setminus X$ there exists an uncountable regular cardinal $\tau_z$ such that the following hold:

1. There exists $T_z \subset X$ homeomorphic to $\tau_z$ such that $z$ is a limit point of $T_z$ in $\beta X$.
2. If $T_z, S_z \subset X$ are homeomorphic to $\tau_z$ and $z$ is a limit point of both $T_z$ and $S_z$ in $\beta X$ then $T_z \cap S_z \neq \emptyset$.

Then $X$ has the fixed filter-to-point property.
Proof. Fix \( f : X \to X \) and \( z \in \beta X \) such that \( \tilde{f}(z) = z \). We may assume that \( z \) is in the remainder. Let \( \tau_z \) and \( T_z \) be as in the hypothesis. Since \( \tilde{f}(z) = z \notin X \), \( f|_{T_z} \) is perfect. By Lemma 3.2, we may assume that \( f|_{T_z} \) is a homeomorphism. By 2 of the hypothesis, \( f(T_z) \cap T_z \) is an unbounded closed subset of \( T_z \). Put \( T = f^{-1}(T_z) \cap T_z \). Then \( T \) and \( f|_T \) satisfy the hypothesis of Lemma 3.3. By Lemma 3.3, \( f \) has a fixed point.

Corollary 3.5. A countably compact GO-space has the fixed filter-to-point property.

This corollary and the main result of the previous sections prompt the following question.

Question 3.6. Let \( X \) be a locally compact subspace of a linearly-ordered space. Does \( X \) have the fixed filter-to-point property?

An assumption of some compactness-type properties is important in this question because of the example of Mazur mentioned in the introduction.

Corollary 3.7. \( \Sigma[0,1]^\tau \) has the fixed filter-to-point property.

The next observation was communicated to the author by several colleagues.

Theorem 3.8. If \( X \times X \) is countably compact and normal, then \( X \) has the fixed filter-to-point property.

Proof. Fix a continuous \( f : X \to X \) such that \( \tilde{f}(z) = z \) for some \( z \in \beta X \setminus X \). Put \( A = \{(x,x) : x \in X\} \) and \( B = \{(x,f(x)) : x \in X\} \). Clearly, \( A \) and \( B \) are closed in \( X \times X \). Since \( X \times X \) is countably compact, \( \beta(X \times X) = \beta X \times \beta X \). Since \( \tilde{f}(z) = z \), \( (z,z) = (z,\tilde{f}(z)) \) is a limit point for both \( B \) and \( A \). Therefore, \( A \) and \( B \) are not functionally separated. Since \( X \times X \) is normal, \( A \) meets \( B \). Pick, \( (x,x) \in A \cap B \). By the definition of \( B \), \( (x,x) = (x,f(x)) \). Thus, \( x \) is a fixed point of \( f \).

In connection with this theorem we would like to mention that M. van Hartskamp and J. van Mill showed [H&M] that there exist a countably compact normal space \( X \) and a fixed-point-free involution \( f : X \to X \) such that \( \tilde{f} \) has a fixed point.

Theorem 3.9. If \( X \) and \( Y \) have the fixed filter-to-point property then so does \( X \oplus Y \).

In connection with this theorem we would like to mention that M. van Hartskamp and J. van Mill showed [H&M] that there exist a countably compact normal space \( X \) and a fixed-point-free involution \( f : X \to X \) such that \( \tilde{f} \) has a fixed point.
Proof. Fix \( f : X \oplus Y \to X \oplus Y \) such that \( \bar{f} \) has a fixed point. We may assume that \( \bar{f}(z) = z \) for some \( z \in \beta X \setminus X \). Put \( S = f^{-1}(Y) \cap X \). Since \( z \in \beta X \setminus X \) is a fixed point of \( \bar{f} \) and \( X \) is clopen in \( X \oplus Y \) we conclude that \( z \) is not a limit point of \( S \). Therefore, \( X \setminus S \) is not empty. Fix an arbitrary \( p \in X \setminus S \). Define \( g : X \to X \) as follows:

\[
g(x) = \begin{cases} f(x) & \text{if } x \notin S \\ p & \text{if } x \in S \end{cases}
\]

Since \( f \) is continuous and \( S \) is clopen in \( X \), \( g \) is continuous. It is clear that \( \tilde{g}(z) = z \). Since \( X \) has the property under investigation there exists \( x \in X \) such that \( g(x) = x \). Since \( p \notin S \) and \( f(S) = \{ p \} \) we conclude that \( g \) does not fix any point in \( S \). Therefore, \( x \notin S \). Since \( x \in X \setminus S \) and \( f \) coincides with \( g \) on \( X \setminus S \), we have \( f(x) = x \).

The next construction of a countably compact space without the property in question will be used to show that our property is not inherited by closed subspaces and is not preserved by perfect maps.

**Example 3.10.** Put \( Y = \langle \omega_1 + 1 \times (\omega_1 + 1) \rangle \times \{0, 1\} \). Let \( Z \) be the quotient space defined by the partition of \( Y \) whose only non-trivial elements are:

1. \( \{\langle \alpha, \beta, 0 \rangle, \langle \omega_1, \alpha, 1 \rangle\}, \text{ where } \alpha < \omega_1 \);
2. \( \{\langle \alpha, \omega_1, 0 \rangle, \langle \alpha, \omega_1, 1 \rangle\}, \text{ where } \alpha < \omega_1 \);
3. \( \infty = \{\langle \omega_1, \omega_1, 0 \rangle, \langle \omega_1, \omega_1, 1 \rangle\} \).

Put \( X = Z \setminus \{\infty\} \). Then \( X \) is countably compact spaces without the fixed filter-to-point property.

Proof. The space \( X \) looks like a transfinite cone with the punctured vertex. Clearly, \( Z = \beta X \) and \( X \) is not normal.

Intuitively, it is clear that if one rotates this cone by 180 degrees around the imaginary axis then no point of \( X \) is fixed by this rotation while the only point in the remainder stays where it is. This explains why \( X \) does not have the property in question. Let us formalize the described map \( f : X \to X \) as follows:

1. \( f(\langle \alpha, \beta, 0 \rangle) = \{\langle \beta, \alpha, 1 \rangle\}, \text{ where } \alpha, \beta < \omega_1 \);
2. \( f(\langle \alpha, \beta, 1 \rangle) = \{\langle \beta, \alpha, 0 \rangle\}, \text{ where } \alpha, \beta < \omega_1 \);
3. \( f(\langle \omega_1, \alpha, 0 \rangle, \langle \omega_1, \alpha, 1 \rangle) = \{\langle \alpha, \omega_1, 0 \rangle, \langle \alpha, \omega_1, 1 \rangle\} \);
4. \( f(\langle \omega_1, \alpha, 0 \rangle, \langle \omega_1, \alpha, 1 \rangle) = \{\langle \omega_1, \alpha, 0 \rangle, \langle \omega_1, \alpha, 1 \rangle\} \).

The map \( f \) is continuous, has no fixed points, and \( \tilde{f}(\infty) = \infty \).
The property is not preserved by continuous perfect maps.

**Example 3.11.** There exist a countably compact space $X$ and its continuous perfect image $Y$ such that $X$ has the fixed filter-to-point property while $Y$ has not.

**Proof.** Put $Z = [\omega_1 + 1 \times (\omega_1 + 1)] \setminus \{\langle \omega_1, \omega_1 \rangle\}$ and $X = Z \times \{0, 1\}$.

It is clear that $X$ admits a continuous perfect (quotient) map onto the space of Example 3.9. Thus, we only need to show that $X$ has the property. By virtue of Theorem 3.10, it suffices to show that $Z$ has the property.

Fix $f : Z \to Z$ such that $f(\omega_1, \omega_1) = \langle \omega_1, \omega_1 \rangle$. Since $f$ maps the remainder of $Z$ into itself, $f$ is perfect. By Lemma 3.2, there exists a closed unbounded $T \subseteq \omega_1$ such that $f$ is a homeomorphism on $D = \{\langle \alpha, \alpha \rangle : \alpha \in T\}$. Notice that any closed set of $Z$ that has $\langle \omega_1, \omega_1 \rangle$ as a complete accumulation point in $\beta Z$ meets either $\{\omega_1\} \times \omega_1$ or $\omega_1 \times \{\omega_1\}$ or the diagonal $\Delta$. Therefore, we may assume that $f(D) \subseteq \Delta$ or $f(D) \subseteq \omega_1 \times \{\omega_1\}$.

If $f(D) \subseteq \Delta$, then by Lemma 3.3, there exists $\alpha$ such that $f(\alpha, \alpha) = \langle \alpha, \alpha \rangle$.

Assume now that $f(D) \subseteq \omega_1 \times \{\omega_1\}$. Since every point of $D$ is of countable character, for every $x \in D$ we can find an open $U_x \subseteq Z$ containing $x$ such that $f(U_x) \subseteq \omega_1 \times \{\omega_1\}$. By continuity, $f(\bigcup_{x \in D} U_x) \subseteq \omega_1 \times \{\omega_1\}$. Since $D$ and $\omega_1 \times \{\omega_1\}$ cannot be separated by disjoint open sets there exists a closed uncountable $T' \subseteq \omega_1$ such that $T' \times \{\omega_1\} \subseteq \bigcup_{x \in D} U_x$. Therefore, $f(T' \times \{\omega_1\}) \subseteq \omega_1 \times \{\omega_1\}$.

By Lemmas 3.2 and 3.3, there exists $\alpha \in \omega_1$ such that $f(\alpha, \omega_1) = \langle \alpha, \omega_1 \rangle$. \qed

Next example shows that the property is not inherited by closed subspaces.

**Example 3.12.** There exist $X$ and a closed subspace $Y$ of $X$ such that $X$ has the fixed filter-to-point property while $Y$ has not.

**Proof.** Let $Y$ be the space of Example 3.9. Recall that $|\beta Y \setminus Y| = 1$ and there exists a closed subset $T = \{x_\alpha : \alpha < \omega_1\}$ of $Y$ that is homeomorphic to $\omega_1$ under correspondence $x_\alpha \leftrightarrow \alpha$ and is convergent to the only point in the remainder.

Put $S = [\omega_2 + 1 \times (\omega_1 + 1)] \setminus \{\langle \omega_2, \omega_1 \rangle\}$. Put $Z = S \oplus Y$. Let $X$ be the quotient space determined by the partition on $Z$ whose only non-trivial elements are $\{\langle \alpha, x_\alpha \rangle, \omega_\beta\}$, where $\alpha < \omega_1$. That is $X$ is obtained by gluing $S$ with $Y$ along sets $T \subseteq Y$ and $\{\omega_2\} \times \omega_1 \subseteq S$. It is clear that $X$ contains a closed copy of $Y$. It is left to show that $X$ has the fixed filter-to-point property.

The remainder $\beta X \setminus X$ has only one point, call it $p$. The set $A = \{\langle \alpha, \omega_1 \rangle : \alpha < \omega_2\}$, which gets in $X$ from $S$, is closed in $X$ and $p$ is the only complete accumulation point for $A$ in $\beta X$. Any other closed set of $X$ homeomorphic to $\omega_2$ meets $A$. By Proposition 3.4, $X$ has the required property. \qed

**Theorem 3.13.** Let $X$ be a countably compact space that has the fixed filter-to-point property and let $C$ be a first-countable compactum. If $\text{Ind}(X) = 0$ or $\text{Ind}(C) = 0$ then $X \times C$ has the fixed filter-to-point property.
Proof. Fix a continuous $f : X \times C \to X \times C$ such that $\tilde{f}$ has a fixed point. Since $X$ is countably compact and $C$ is compact, $\beta(X \times C) = \beta X \times C$. We may assume that $\tilde{f}$ fixes $\langle x^*, c^* \rangle$, where $x^* \in \beta X \setminus X$ and $c^* \in C$. By first-countability there exists a family $\{S_n : n \in \omega\}$ with the following properties:

P1: $S_n \subset C \setminus \{c^*\}$ and $S_n$ is closed in $C$;

P2: $\{S_n : n \in \omega\}$ is locally finite in $\bigcup_n S_n$;

P3: $C \setminus \{c^*\} = \bigcup_n S_n$.

Put $X^* = X \times \{c^*\}$ and $A_n = f^{-1}(X \times S_n) \cap X^*$. Let us list the properties of $A_n$'s to be used later.

P4: $A_n$ is closed in $X^*$. This follows from P1 and continuity of $f$.

P5: $\{A_n : n \in \omega\}$ is locally finite in $\bigcup_n A_n$. This follows from P2.

Define $g : X^* \to X^*$ as follows: $g(x, c^*) = (y, c^*)$, where $f(x, c^*) = (y, c)$ for some $c \in C$. That is, $g$ is the composition of $f|_{X^*}$ and the projection onto $X^*$. The definition of $g$ implies that

P6: $g(z) = f(z)$ for all $z \in X^* \setminus [\bigcup_n A_n]$ and $\tilde{g}(x^*, c^*) = \langle x^*, c^* \rangle$.

Let $F$ be the set of all elements of $X^*$ that are fixed by $g$. By continuity, $F$ is closed. By P4, $F_n = F \cap A_n$ is closed too. By P5,

P7: $\{F_n : n \in \omega\}$ is locally finite in $\bigcup_n A_n$.

We have three cases to consider.

Assume that $\bigcup_n F_n$ is not closed in $X^*$. Fix $p \in [\bigcup_n F_n] \setminus [\bigcup_n F_n]$. By P7, $p \in X^* \setminus [\bigcup_n A_n]$. By P6, $f(p) = g(p)$. Since $p \in F$, $g(p) = p$. Thus, $f$ fixes $p$.

Now we assume that $\bigcup_n F_n$ is closed in $X^*$ and $F_n \neq \emptyset$ for finitely many indices only. Since $\langle x^*, c^* \rangle$ is fixed by $\tilde{f}$ and $f(F_n) \subset X \times S_n$, we conclude that $\langle x^*, c^* \rangle$ is not a limit point of $f(F_n)$, and therefore, for $F_n$. Since only finitely many $F_n$'s are non-empty, $\langle x^*, c^* \rangle$ is not a limit point of $\bigcup_n F_n$ either.

We claim that there exists $U$ clopen in $X^*$ such that $\langle x^*, c^* \rangle$ is a limit point of $U$ and $\bigcup_n F_n \subset X^* \setminus U$. Indeed, if $Ind(X) = 0$ then such a $U$ clearly exists. If $Ind(C) = 0$, let $W$ be a clopen subset of $C$ that contains $c^*$ and misses $\bigcup \{S_n : F_n \neq \emptyset\}$. Then $U = f^{-1}(W) \cap X^*$ is as desired. Fix $p \in U$. Define $h : X^* \to X^*$ as follows:

$$h(z) = \begin{cases} g(z) & \text{if } z \in U \\ p & \text{if } z \in X^* \setminus U \end{cases}$$
Since $U$ is clopen $h$ is continuous. Since $(x^*, c^*)$ is not a limit point of $X^* \setminus U$ and $h|_U = g|_U$ we have $\tilde{h}(x^*, c^*) = \tilde{g}(x^*, c^*) = (x^*, c^*)$. Since $X^*$ has the fixed filter-to-point property, $h(p^*) = p^*$ for some $p^* \in X^*$. Since $p \in U$ and all points in the complement of $U$ are mapped to $p$, we conclude that $h$ does not fix any points in $X^* \setminus U$. Therefore, $p^*$ is in $U$. Since all points fixed by $g$ are in $F$ and $F \cap A_n = F_n \subset X^* \setminus U$, we conclude that $p^* \notin \bigcup_n A_n$. By P6, $h(p^*) = f(p^*)$. Thus, $f$ fixes $p^*$.

Remark 3.14. Observe that the only conclusion we made from first-countability of $C$ is that $C \setminus \{c^*\}$ can be covered by closed subsets of $C$ that lie in $C \setminus \{c^*\}$ and form a locally finite family in $C \setminus \{c^*\}$. Therefore, the requirement "$C$ is first-countable" can be replaced, in particular, by "$C \setminus \{c\}$ is paracompact for every $c \in C$". The author does not know whether zero-dimensionality or countable compactness can be dropped from the conditions of the above theorem.

This remark implies, in particular, that if a countably compact space $X$ with the fixed filter-to-point property is multiplied by the one-point compactification of a discrete space then the result has the fixed filter-to-point property.

We would like to finish this work with a general problem that may lead to finding of new classes of spaces with the property under consideration.

Problem 3.15. Let $X$ and $Y$ have the fixed filter-to-point property. What additional conditions on $X$ and/or $Y$ guarantee that $X \times Y$ has the fixed filter-to-point property?

References


